Multiple representations in mathematical practice: Cluster algebras as a case study*

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Abstract. This paper investigates the role played by different modes of representations both in the discovery of new objects of study in mathematical practice and in facilitating understanding, using examples taken from cluster algebras as a case study. Cluster algebras arose at the turn of the 21st century through work by S. Fomin and A. Zelevinsky [2] with the aim of providing an algebraic and combinatorial framework for studying dual canonical bases, which had previously been defined geometrically by G. Lusztig in the early 1990s. I use this case study to argue that the availability of multiple ways of expressing mathematical content can bring to light new classes of mathematical objects which are seen as highly salient from a practice point of view. In such situations, examining the choice of one form of representation over another can also yield insights into how such expressions support various different aims, such as building familiarity with certain concepts or studying connections between diverse mathematical theories and domains.

Keywords: Notations · Discovery · Understanding · Multiple representations · Mathematical practice · Cluster algebras · Quivers · Matrices

Studying the use of signs (e.g. notations or figures) in mathematical practice can help us to better understand how the computational and cognitive constraints of humans as finite agents shape our mathematical theories [5]. Indeed, the choice of a notational system is significant for how well it is suited for a given task [3]. One of the roles for visual representations which has been studied in recent literature is how they may support the emergence of new mathematical concepts and results [1, 4].

In this paper, I examine some of the visual representations appearing in the context of cluster algebras (described in further detail below) not only in terms of their fruitfulness, but also for the role they play in facilitating certain kinds of tasks. In tasks which are commonly performed with paper and pencil (or tablet and stylus), there is an apparent preference for choosing certain kinds of representation over others by the mathematicians working on cluster algebras. This paper investigates the specific features of different representations in relation to what is required by particular tasks.

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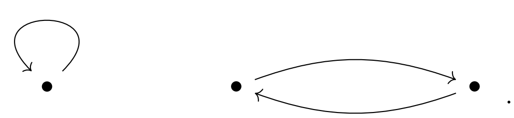
Cluster algebras were introduced into the mathematical literature by S. Fomin and A. Zelevinsky [2] in order to study connections between dual canonical bases and total positivity, which had previously been investigated by G. Lusztig in the early 1990s. From the outset, cluster algebras were intended to give an algebraic and combinatorial interpretation of dual canonical bases which Lusztig had defined geometrically. In present day, cluster algebras constitute an especially active area of research, with the paper by Fomin and Zelevinsky [2] that first introduced the notion of cluster algebras into the literature having received more than 2300 citations since its publication. Significantly, numerous 'deep connections' quickly arose between cluster algebras and other areas of mathematics—including quiver representations, Poisson geometry, Teichmüller theory, group theory, integrable systems, and mathematical physics.

A cluster algebra is a commutative ring which is defined over an ambient field $\mathcal{F}$ of rational functions in n variables and which enjoys a particular combinatorial structure. What sets a cluster algebra apart from other sorts of commutative rings is that "a cluster algebra is not presented at the outset via a complete set of generators and relations" ([6], p. 1) but instead constructed from some initial data—contained in a so-called *seed*—which defines the cluster algebra. A 'seed' contains a *cluster*, i.e., a set of distinguished generators called *cluster variables*, along with an *exchange matrix*, which encodes the rules for obtaining new cluster variables from previous ones via an iterative procedure called *mutation*. Shortly after the introduction of cluster algebras, it was found that the rules governing mutation can be encoded not only in (certain kinds of) matrices, but also in *quivers*, which are directed graphs but viewed from a new perspective.

Let us consider mutation in the context of quivers. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where the elements $i \in Q_0$ are called *vertices*; the elements $\alpha \in Q_1$ are called *arrows*; and the maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ are called the *source* and *target* functions, respectively. For each arrow $\alpha \in Q_1$, $s(\alpha)$ gives us the vertex $i \in Q_0$ which is the starting point of α. Similarly, $t(\alpha)$ gives us the vertex $j \in Q_0$ where α terminates. A *loop* is an arrow α such that $s(\alpha) = t(\alpha)$. A 2-*cycle* is a pair of mutually distinct arrows α and β where $s(\alpha) = t(\beta)$ and $s(\beta) = t(\alpha)$. Graphically, we have

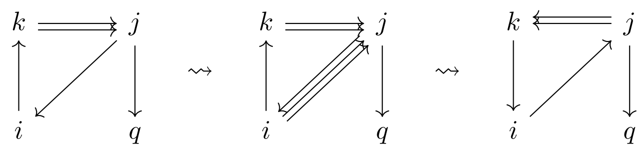


Loop 2-cycle

Consider a finite quiver Q without any loops or 2-cycles. *Mutation* $\mu_k(Q)$ *of* Q *in direction* $k \in Q_0$ is performed according to the following procedure:

(a) For each subquiver $i \to k \to j$ in Q, add a new arrow $i \to j$.
(b) Reverse the direction of each arrow α in which either the source or target function coincides with k, i.e., $s(\alpha) = k$ or $t(\alpha) = k$.
(c) Remove both arrows of any 2-cycles which may have arisen in step (a).

This procedure may look something like the following in graphical form:



Cluster algebras that are defined using quivers belong to a particular class, which corresponds to a special case of the general setting introduced in [2] that makes use of skew-symmetrizable matrices for defining cluster algebras. A square integer matrix $B = (b_{ij})$ is *skew-symmetrizable* if there are positive integers d_l such that $d_i b_{ij} = -d_j b_{ji}$ for all i and j. Matrix mutation proceeds according to certain rules defined for changing the entries of the matrix. We can associate a *signed adjacency matrix* $B(Q) = (b_{ij})$ to a quiver Q defined as above. $B(Q)$ is a skew-symmetric matrix in which $b_{ij} = -b_{ji} = \ell$, where ℓ is the number of arrows from vertex i to vertex j. Mutating $B(Q)$ corresponds to mutating Q, which we can observe in our specific example if, by a common abuse of notation, we allow $k = 1$, $j = 2$, $i = 3$, and $q = 4$:

$$\begin{bmatrix} 0 & 2 & -1 & 0 \\ -2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -2 & 1 & 0 \\ 2 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

For certain tasks which involve computation performed “by hand” (i.e. with pencil and paper), such as mutation, it seems to be the case that cluster algebra theorists often favor using quivers over using matrices. Although these different forms of representations are intended to encode the *same* information in their role in cluster algebras, and are from this perspective seen as equivalent, the differences in their visual appearances is striking. Close examination of the visual features of quivers and matrices, respectively, in relation to computation and other various tasks, may yield some insights into the importance of representations in mathematical practice more broadly.

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