A Compositional Semantics for Venn Diagrams

Bryan Pickel[1] and Brian Rabern[2]

[1] University of Glasgow, UK
bryan.pickel@glasgow.ac.uk
[2] University of Edinburgh, UK
brian.rabern@ed.ac.uk

The important work of [3], [2], and others has applied techniques for analyzing formal languages to Venn Diagrams. This work has shown that Venn Diagrams can be specified by recursive formation rules, given a model-theoretic semantics, and a complete proof theory. But these theories are not rule-to-rule compositional. We will develop a rule-to-rule compositional semantics for Venn Diagrams. This endeavor sheds light on whether Venn diagrams present a distinct mode of representation compared to language, and the extent to which ”the elements and combinatorial rules for [Venn] diagrams are very different than those for sentences” ([1]: 153-4).

Our investigation delineates two methods for syntactically deriving a Venn Diagram. The *cell-first* approach first constructs the cells or minimal regions of a diagram out of its basic regions. Notice that diagram 1 has eight cells, *a*–*h*.

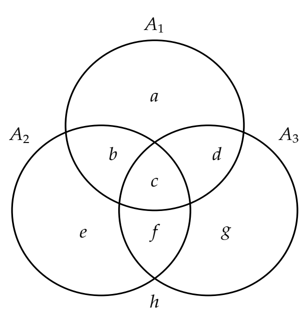


Diagram 1: Undecorated Venn diagram with labeled cells

The cell-first approach then adds markers to indicate that the region represents a non-empty or empty set. The *region-first approach*, by contrast, directly constructs non-basic regions and adds markers to indicate that the region represents an empty or non-empty set. Thus, to indicate that the shaded region represents an empty set, the former would construct Diagram 2 in two steps by shading cell b and then cell c, while the latter would directly shade the region $b \cup c$ to indicate that it represents an empty set. Likewise for “salvation”, such as in Diagram 3, where hatch marks indicate that the region represents a non-empty set.

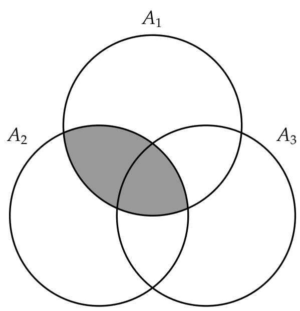


Diagram 2: A diagram with region $b \cup c$ destroyed

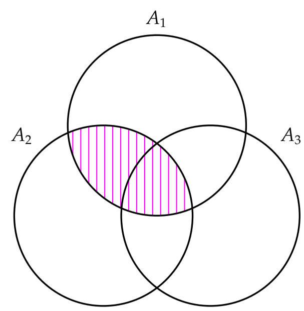
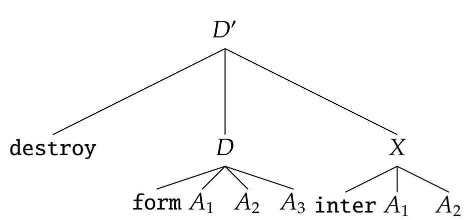


Diagram 3: A diagram with region $b \cup d$ saved

In this talk, we focus on the region-first approach. Let a diagram $D = \langle B, \Delta, \Sigma \rangle$ where B is the set of basic regions; Δ is the set of destroyed regions; and Σ is the set of saved regions. Venn Diagrams can be constructed via the following syntactic construction rules:

Basic Regions: $\{A_1, A_2, \ldots, A_i, \ldots\}$
Form Rule: If $X_1, \ldots X_n$ are basic regions satisfying the mutual overlap constraint, then $\texttt{form}(X_1, \ldots, X_n)$ is a diagram with regions including $X_1, \ldots, X_n$ and R.
Intersection Rule: If X_1 and X_2 are regions of D, then $\texttt{inter}(X_1, X_2) = \bigcap\{X_1, X_2\}$ is a region of D, provided that it is non-empty.
Union Rule: If X_1 and X_2 are regions of D, then $\texttt{union}(X_1, X_2) = \bigcup\{X_1, X_2\}$ is a region of D, provided that it is non-empty.
Complement Rule: If X_1 and X_2 are regions of D, then $\texttt{comp}(X_1, X_2) = X_1 - X_2$ is a region of D, provided that it is non-empty.
Destruction Rule: If D is a diagram and if X is a region of D, then $\texttt{destroy}(D, X)$ is a diagram.
Salvation Rule: If D is a diagram and if X is a region of D, then $\texttt{save}(D, X)$ is a diagram.

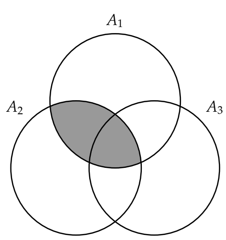
For example this is the syntactic derivation of Diagram 2:



That's the syntax.[3] Now the clauses for the compositional semantics.

- Semantics for Regions
  - For any basic region X, $[\![X]\!]_{\mathfrak{A}} = I(X)$, where $\mathfrak{A} = \langle U, I \rangle$.
  - $[\![\texttt{inter}(X_1, X_2)]\!]_{\mathfrak{A}} = [\![X_1]\!]_{\mathfrak{A}} \cap [\![X_2]\!]_{\mathfrak{A}}$
  - $[\![\texttt{union}(X_1, X_2)]\!]_{\mathfrak{A}} = [\![X_1]\!]_{\mathfrak{A}} \cup [\![X_2]\!]_{\mathfrak{A}}$
  - $[\![\texttt{comp}(X_1, X_2)]\!]_{\mathfrak{A}} = [\![X_1]\!]_{\mathfrak{A}} - [\![X_2]\!]_{\mathfrak{A}}$
- Semantics for Diagrams
  - $[\![\texttt{form}(X_1, \ldots, X_n)]\!]_{\mathfrak{A}} = 1$.
  - $[\![\texttt{destroy}(D, X)]\!]_{\mathfrak{A}} = \begin{cases} 1, \textit{if } [\![D]\!]_{\mathfrak{A}} = 1 \textit{ and } [\![X]\!]_{\mathfrak{A}} = \emptyset, \\ 0, \textit{otherwise.} \end{cases}$
  - $[\![\texttt{save}(D, X)]\!]_{\mathfrak{A}} = \begin{cases} 1, \textit{if } [\![D]\!]_{\mathfrak{A}} = 1 \textit{ and } [\![X]\!]_{\mathfrak{A}} \neq \emptyset, \\ 0, \textit{otherwise.} \end{cases}$

The truth-conditions of a diagram such as Diagram 2 = $\texttt{destroy}\big(\texttt{form}(A_1, A_2, A_3), \texttt{inter}(A_1, A_2)\big)$ can then be derived as follows.



$$
\begin{aligned}
[\![\ \text{(diagram)}\ ]\!]_{\mathfrak{A}} = 1 \quad & \text{iff } \big[\!\big[\texttt{destroy}\big(\texttt{form}(A_1, A_2, A_3), \texttt{inter}(A_1, A_2)\big)\big]\!\big]_{\mathfrak{A}} = 1 \\
& \text{iff } \big[\!\big[\texttt{form}(A_1, A_2, A_3)\big]\!\big]_{\mathfrak{A}} = 1 \text{ and } \big[\!\big[\texttt{inter}(A_1, A_2)\big]\!\big]_{\mathfrak{A}} = \emptyset \\
& \text{iff } 1 = 1 \text{ and } \big([\![A_1]\!]_{\mathfrak{A}} \cap [\![A_2]\!]_{\mathfrak{A}}\big) = \emptyset \\
& \text{iff } I(A_1) \cap I(A_2) = \emptyset
\end{aligned}
$$

References

1. Camp, E.: Thinking with maps. Philosophical Perspectives **21**(1), 145–182 (2007). https://doi.org/10.1111/j.1520-8583.2007.00124.x
2. Hammer, E., Danner, N.: Towards a model theory of diagrams. Journal of Philosophical Logic **25**(5), 463–482 (1996). https://doi.org/10.1007/BF00257381
3. Shin, S.J.: The logical status of diagrams. Cambridge University Press (1994)

[3] Provided a definition of the *mutual overlap constraint*. A (non-empty) set of basic regions $\{X_1, \ldots, X_n\}$ of the plane R meets the mutual overlap constraint iff for all Γ: if $\Gamma \subseteq \{X_1, \ldots, X_n\}$, then $\bigcap\big(\Gamma \cup \{R\}\big) - \bigcup\big(\{X_1, \ldots, X_n\} - \Gamma\big) \neq \emptyset$.